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THE FOUR COLOR PROBLEM.[*]

BY PHILIP FRANKLIN.

1. By a map we shall understand a subdivision of an inversion plane or sphere by means of a finite number of circular arcs into a finite number of regions, which completely cover it. There is no loss of generality in this restriction, as a "map" on any surface of genus zero, with a finite number of regions bounded by simple curves, may be deformed into a map of the type just described. A *side* is a line along which two distinct regions touch each other; a *vertex* is a point which belongs to three or more regions. The problem of coloring a map with a given number of colors (denoted in what follows by A, B, C, etc.) is the association of a color with each region in such a way that any two regions with a side in common are given different colors. Two regions with only a vertex (or a finite number of vertices) in common may of course have the same color.

Whether every map can be colored with four colors is an outstanding question, for while no map has ever been exhibited which could not be colored with four colors, no rigorous demonstration of the possibility for the general case has ever been given.[†] It is known that four colors are necessary to color some maps and that five colors are sufficient to color all maps. If any maps which can not be colored in four colors exist, there must be one such map of a minimum number of regions. We will call such a map an *irreducible* map. It is known that an irreducible map has the following properties:[‡]

1. Each vertex belongs to three and only three regions.
2. No group of less than five regions forms a multiply-connected portion of the map. (Consequently there are no two-, three- or four-sided regions and no multiply-connected regions.)
3. No group of five regions forms a multiply-connected portion of the map unless the group consists of the five regions surrounding a pentagon.
4. No edge is surrounded by four pentagons.
5. No region is completely surrounded by pentagons.

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[*] Presented to the National Academy of Sciences, November 17, 1920.

[†] A history of the question, with a bibliography, is given in the thesis of Alfred Errera "Du Coloriage des Cartes, etc.," Brussels, 1921.

[‡] For an account of these reductions of the problem, which are due to A. B. Kempe and G. D. Birkhoff, see a paper by the latter "The Reducibility of Maps," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), p. 116.

6. No even-sided region is completely surrounded by hexagons. Each of these statements amounts to saying that a certain configuration is not possible in an irreducible map. Such a configuration will be called a *reducible* configuration.

In this paper we shall derive a few additional reducible configurations, by means of which it can be shown that the number of regions in an irreducible map is greater than 25. An example is also given of a map of 42 regions which, although colorable in four colors, satisfies all the conditions which have been derived for an irreducible map. It may be taken as showing the extreme lack of generality of the results thus far obtained for this problem.

2. It is interesting to find out some of the properties of a map not containing any region of less than five sides, as this is a property of an irreducible map. Since our map is drawn on a sphere, the Euler formula (applied to a manifold of genus zero) gives:

$$a_0 - a_1 + a_2 = 2, \tag{1}$$

where a_0, a_1, a_2 are the number of vertices, sides and regions respectively. Also since only three regions touch any one vertex, we have:

$$2a_1 = 3a_0 = \sum_5 \nu A_\nu, \tag{2}$$

where A_ν means the number of regions of ν sides in the map; the last two expressions are each equal to the first since they represent twice the number of sides in the map, counted first with reference to vertices, then with reference to regions. From (1) and (2) we obtain:

$$a_1 = 3(a_2 - 2), \qquad a_0 = 2(a_2 - 2). \tag{3}$$

From (2), (3) and the fact that $a_2 = \sum_5 {}_5A_\nu$, we see that

$$6(\sum_5 A_\nu - 2) = \sum_5 \nu A_\nu. \tag{4}$$

This may be written:

$$A_5 = 12 + \sum_7 (\nu - 6)A_\nu \tag{5}$$

and since the second term on the right is positive, A_5 must be at least 12, and we have the well-known theorem:

Every map containing no triangles or quadrilaterals and having three regions abutting on each vertex contains at least twelve pentagons.[*]

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[*] Cf. Kempe, A. B., "The Geographical Problem of the Four Colors," AMERICAN JOURNAL OF MATHEMATICS, Vol. II (1879), p. 198.

We shall also prove that such a map must contain either:

A pentagon adjacent to two other pentagons,
A pentagon adjacent to a pentagon and to a hexagon, or
*A pentagon adjacent to two hexagons.**

For, consider a map with none of these combinations of regions and let us count the number of vertices in the map which belong to a hexagon or a pentagon. We find that the number of vertices contributed by hexagons nowhere in contact with pentagons will be greater than twice the number of such hexagons since each hexagon has six vertices and no vertex belongs to more than three hexagons; pentagons isolated from hexagons or other pentagons will give five vertices each; two pentagons adjacent to each other but to no other pentagons or hexagons give eight vertices together, and hence average four each; while a pentagon adjacent to a hexagon gives over four vertices, since of its five vertices we need only deduct two thirds to account for the two where the hexagon joins it. Thus if none of the three conditions enumerated above existed, the number of vertices would be at least $4A_5 + 2A_6$. That is we would have to have:

$$a_0 \geq 4A_5 + 2A_6. \tag{6}$$

But from (5) and the obvious inequality:

$$0 \geq \sum_7 (7 - \nu)A_\nu \tag{7}$$

there results:

$$\sum_7 A_\nu + 12 \leq A_5 \tag{8}$$

or

$$\sum_5 A_\nu + 12 \leq 2A_5 + A_6 \tag{9}$$

and since (from (3)):

$$\sum_5 A_\nu = a_2 = a_0/2 + 2, \tag{10}$$

$$a_0/2 + 14 \leq 2A_5 + A_6, \tag{11}$$

$$a_0 + 28 \leq 4A_5 + 2A_6, \tag{12}$$

which contradicts (6) and thus proves the theorem.

The above theorem is not restricted to irreducible maps, but it follows from it that if the configurations there shown to be present were reducible there could not be any irreducible maps and the four-color problem would be solved. While it does not appear to be possible to prove this, there are a number of more complicated configurations which are reducible.

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* That every such map contains either two adjacent pentagons or a pentagon adjacent to a hexagon was proved by Wernicke, P., "Über den Kartographischen Vierfarbensatz, Mathematische Annalen," Vol. 58 (1904), p. 419.

3. To obtain these configurations, and prove their reducibility, we shall need the notion of *chains*, originated by Kempe,* and employed by Birkhoff.* If a map is colored, or partially colored, a group of regions colored in two colors (say *A* and *B*), forming a connected region, and such that each region adjacent to a region of the group is either colored in one of the remaining two colors (*C* or *D*) or not yet colored, is said to form a *chain* (an *AB* chain). Evidently we may obtain a second coloration or partial coloration of the map by interchanging the two colors on a single chain, and unless the map contains only one chain in this pair of colors, the new coloration will differ from the old by more than a mere permutation of the colors of the whole map.

Furthermore since two chains with no color in common, as an *AB* chain and a *CD* chain, can not "cross" each other, if we have a closed circuit consisting of an *AB* chain, or an *AB* chain and an uncolored region in the case of a partially colored map, it follows that the *C* and *D* regions on one side of the closed circuit can not belong to the same *CD* chain as those on the opposite side of the circuit. Thus in Fig. 1, if 1 is an uncolored region,

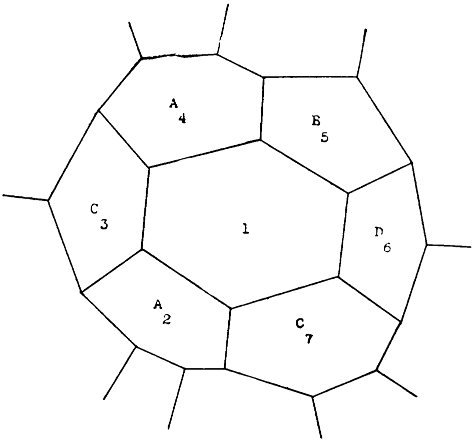

FIG. 1.

and 2 and 4 are joined by an *AB* chain, 3 must belong to a *CD* chain distinct from the one containing 6 and 7. Consequently we may interchange the colors in the *CD* chain containing 3 without affecting 6 and 7. Since, in most of the applications of this process, we shall only be concerned with the arrangement of the colors about the uncolored region, and the rest of them are unchanged by this operation, we shall briefly refer to this operation as "changing 3 to a *D*." The value of these operations will be seen in the proofs which follow.

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* L. c.

4. We shall now prove that

A side of a hexagon surrounded by this hexagon and three pentagons is a reducible configuration. For, if it were present in an irreducible map, and we erased the dotted lines as in Fig. 2, we would obtain a new map which would contain fewer regions than an irreducible map and hence be colorable.

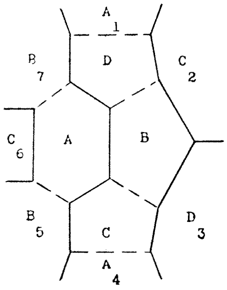

FIG. 2 *a*.

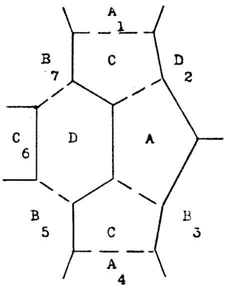

FIG. 2 *b*.

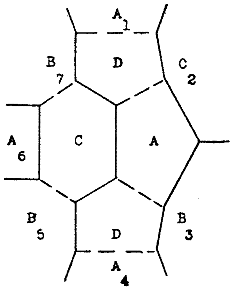

FIG. 2 *c*.

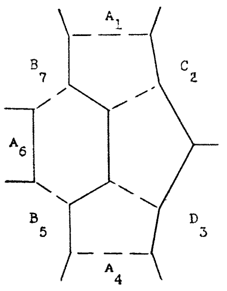

FIG. 2 *d*.

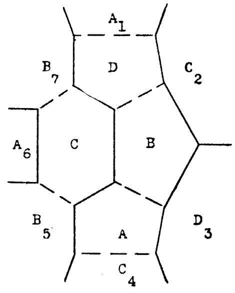

FIG. 2 *e*.

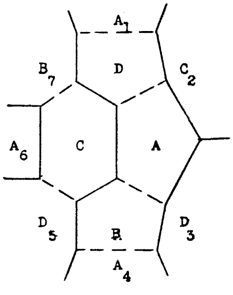

FIG. 2 *f*.

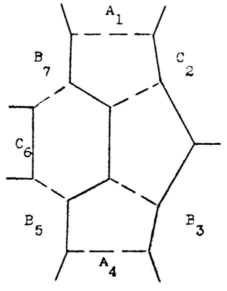

FIG. 2 *g*.

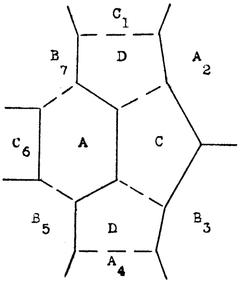

FIG. 2 *h*.

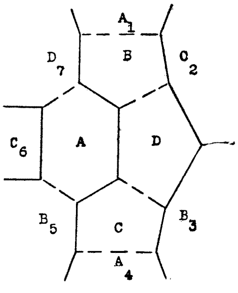

FIG. 2 *i*.

From the way we selected the lines which were erased, regions 1 and 4 would have the same color (say A) while regions 5 and 7 would have a different common color (say B). Of the five essentially distinct colorations, the three cases shown in 2 *a*, 2 *b* and 2 *c* permit of immediate coloration, as indicated. In the case shown in 2 *d*, if 5 is joined to 7 by a BD chain, 6 may be changed to a C, reducing the problem to case 2 *a*, while if 5 is joined to 3 by a *BD* chain, 4 may be changed to a *C*, and the map colored as shown in 2 *e*. If neither of these chains exist, 5 may be changed to a *D*, and the map colored as shown in 2 *f*. Finally in the case given in 2 *g*, either a *BD* chain joins 7 with 5, and we reduce to 2 *c* by changing 6 to *A*; or a *BD* chain joins 7 with 3, and we color as in 2 *h* after interchanging *A* and *C* in the *AC* chain including 1 and 2; or 7 may be changed to a *D* and we color as in 2 *i*.

If 5 and 7 had a side in common in our original map, we could not erase the dotted lines, and still leave a map; but in this case we would have three regions forming a multiply-connected piece. If 1 and 4 had a side in common, we would have a group of five regions forming a multiply-connected region of the map, and not all adjacent to the same pentagon. Hence both these cases are excluded by the properties of irreducible maps given in the first section.

If a pentagon is in contact with three pentagons, a hexagon, and a fifth region of any number of sides, either the hexagon is adjacent to this fifth region, in which case the three adjacent pentagons, with the initial pentagon, completely surround an edge, or the hexagon is adjacent to two pentagons, and we have a side of the hexagon completely surrounded by this hexagon and three pentagons. In either case, it is reducible and we have the result:

A pentagon in contact with three pentagons and a hexagon is a reducible configuration.

Also if a pentagon is in contact with two pentagons and three hexagons, if the two pentagons are not adjacent, they are separated by a hexagon, which with the three pentagons forms a reducible configuration. If the two pentagons are adjacent, we proceed as follows: We erase the boundaries which are dotted (Fig. 3 *a*) and color the resulting map. If all the regions

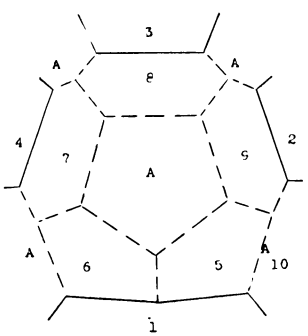

FIG. 3 *a*.

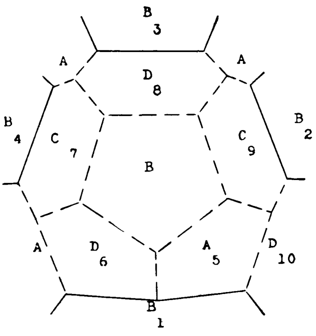

FIG. 3 *b*.

1, 2, 3, 4 are not colored in one color, there are two of them, say 1 and 2, which are separated by a single region and in different colors, say B and C respectively. We then color 5 with C, and 6, 7, 8, 9 in turn, which is possible since taking them in this order we shall never come to a region surrounded by more than three different colors. On the other hand, if all the regions 1, 2, 3, 4 are colored in the same color, say B, either there is no BC chain joining all these regions, in which case we can change some of these regions to C and reduce our problem to the case just discussed, or the AD chain containing the region 10 is separated from the other regions marked A and can be changed to a D. The map is then colored as shown in Fig. 3 *b*. This proves the theorem:

A pentagon surrounded by two pentagons and three hexagons is a reducible configuration.

In this proof we have omitted any reference to the case where the dotted lines can not be erased without giving rise to a region which meets itself along one edge. We shall also do this in future cases where, as in this case, it may be excluded by the considerations used for this purpose in the proof of our first theorem.

By a method quite similar to the above, we could easily show that any odd-sided region, completely surrounded by one or more pairs of pentagons, the two of each pair being adjacent, and a number (necessarily odd) of hexagons, is a reducible configuration.

To lead up to a slightly more general theorem, we repeat Birkhoff's proof of the reducibility of an even-sided region surrounded by hexagons, for definiteness stating the proof for a hexagon so surrounded. We erase the dotted lines of Fig. 4 *a* and obtain the coloration shown. If all the

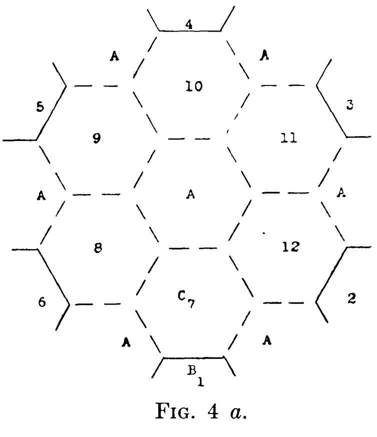

FIG. 4 *a*.

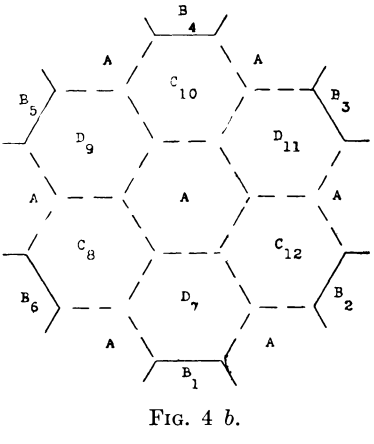

FIG. 4 *b*.

regions 1, 2, 3, 4, 5, 6 are not colored in one color, there are two of them, say 1 and 2, which are separated by a single region and in different colors, say *B* and *C*. We then color 7 with *C*, and color 8, 9, 10, 11, 12 in turn, which is possible since we shall find each adjacent to regions of three different colors at most, and thus have a fourth with which to color each. If 1, 2, 3, 4, 5, 6 are all in the same color, our map is colored as in Fig. 4 *b*.

Our generalization is to the case where two adjacent hexagons are replaced by pentagons, and the above method is directly applicable, provided we imagine one of the regions marked *A* as shrunk to a point. This shows that:

An even-sided region completely surrounded by hexagons and pairs of pentagons, the two of each pair being adjacent, is a reducible configuration.

If a hexagon is surrounded by two hexagons and four pentagons, either the pentagons are grouped so as to come under the theorem just proved, or one of the edges of the hexagon is in contact with three pentagons which we proved above was a reducible configuration. Thus:

A hexagon surrounded by four pentagons and two hexagons is a reducible configuration.

A region of an even number of sides ($2n$) *surrounded by* $2n - 2$ *pentagons and two other regions, which are adjacent, is reducible.*

To fix the ideas, we state the proof for an octagon. We erase the dotted lines (Fig. 5) and color the resulting map. If 4 is a *C*, we color 15, 14, 13, 12, 11, 10 in turn which will be possible since each will only be adjacent to regions in at most three different colors when we come to it. If 4 is a *B* or *D*, we color 10 *D* or *B* respectively, and color 11 *C*. If 6 is a *C*, we color 15, 14, 13, 12 in turn, as before; while if it is *B* or *D* we color 12 *D* or *B* respectively and 13 *C*. We then color 14 and 15 *D*, *C*; *B*, *D*; or *B*, *C*; according as 8 is *B*, *C* or *D*.

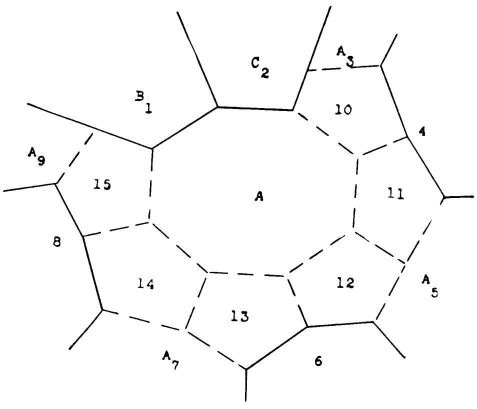

FIG. 5.

A region of an odd number of sides ($2n - 1$) *surrounded by* $2n - 2$ *pentagons and one other region is reducible.*

We give the proof for a heptagon. After erasing the dotted lines (Fig. 6) we color the new map, as indicated. Reasoning exactly as for the preceding theorem, we show that if 5 is a C the map is colorable, while if it is not C we color 11 and 12, giving 12 the color C. Then we color 13 and 14 D, C; B, D; or B, C; according as 7 is B, C or D.

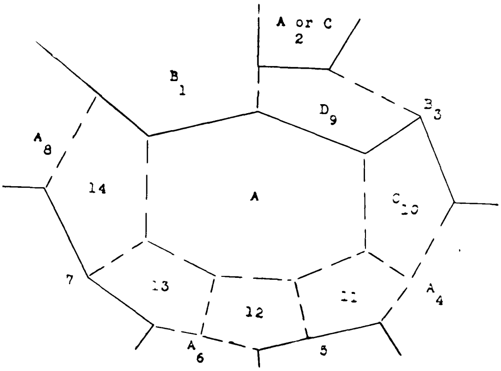

FIG. 6.

It follows from our last two theorems that:

An n-gon in contact with n — 1 pentagons is reducible.

5. We will now deduce certain inequalities which must be satisfied by the numbers A_ν if the map is an irreducible one. We have shown that such a map must not contain:

(*a*) An n-gon in contact with $n - 1$ pentagons,
(*b*) A pentagon in contact with three pentagons and one hexagon,
(*c*) A pentagon in contact with two pentagons and three hexagons,
(*d*) A hexagon in contact with four pentagons and two hexagons,

in addition to the six configurations given in the first section, and have also shown that the equation (see (5)):

$$A_5 = 12 + \sum_{7} (\nu - 6)A_\nu \tag{13}$$

applies to such a map.

From (*a*) we know that every region of our map is in contact with at least *two* regions of more than five sides. Hence the number of sides (a side being counted with each of the two regions it separates) of regions with more than five sides must be at least equal to twice the total number of regions. That is:

$$\sum_{6} \nu A_\nu \geq 2 \sum_{5} A_\nu. \tag{14}$$

To obtain a second inequality from the remaining conditions, we write:

A_5^0 = the number of pentagons in contact with no region of more than six sides,
A_5^1 = the number of pentagons in contact with only one such region,
A_5^2 = the number of pentagons in contact with two or more such regions,
A_6^0 = the number of hexagons in contact with no such regions, and
A_6^1 = the number of hexagons in contact with at least one such region.

From (a) it follows that each region A_5^2, A_6^1 (we thus abbreviate regions of the type counted in A_5^2, A_6^1) as well as those of more than six sides is in contact with at least *two* regions of more than five sides. Also from (a), (b) and (c) it follows that each pentagon A_5^0 is in contact with at least *four* hexagons, and from (a) and (b) that each pentagon A_5^1 is in contact with at least two hexagons in addition to the one region of more than six sides; and therefore to at least *three* regions of more than five sides. Finally from (a) and (d) we see that each hexagon A_6^0 is in contact with at least *three* other hexagons. Thus we have:

$$\sum_6 \nu A_\nu \geq 4A_5^0 + 3A_5^1 + 2A_5^2 + 3A_6^0 + 2A_6^1 + 2\sum_7 A_\nu. \tag{15}$$

But from the definitions of the regions A_5^1, etc.:

$$\sum_7 \nu A_\nu \geq A_5^1 + 2A_5^2 + A_6^1. \tag{16}$$

If we add corresponding members of (15) and (16), recollecting that $A_5 = A_5^0 + A_5^1 + A_5^2$ and $A_6 = A_6^0 + A_6^1$, we obtain the result:

$$\sum_6 \nu A_\nu + \sum_7 \nu A_\nu \geq 4A_5 + 3A_6 + 2\sum_7 A_\nu, \tag{17}$$

which may be written:

$$2\sum_7 \nu A_\nu \geq 4A_5 - 3A_6 + 2\sum_7 A_\nu. \tag{18}$$

For a map which contains no regions of more than seven sides, we may obtain a somewhat stronger inequality, by using:

$$4A_6 + 5A_7 \geq 4A_5^0 + 3A_5^1 + 2A_5^2, \tag{19}$$

$$5A_7 \geq A_5^1 + 2A_5^2, \tag{20}$$

which are analogous to (15) and (16); except that by considering only sides of pentagons in contact with regions of more than five sides we are enabled to use (a) in deriving the left members. They give:

$$4A_6 + 10A_7 \geq 4A_5, \tag{21}$$

which is only applicable to maps composed entirely of pentagons, hexagons, and heptagons.

By using (13) we may reduce (14) and (18) to the respective forms:

$$\sum_{6} (10 - \nu)A_\nu \geq 24, \tag{22}$$

$$2 \sum_{7} (11 - \nu)A_\nu \geq 48 - 3A_6. \tag{23}$$

These two inequalities show that the map we are considering must have at least 25 regions, and if it have only 25, they must be 17 pentagons, 3 hexagons, and 5 heptagons. For if the map contained two hexagons (22) would give:

$$\sum_{7} (10 - \nu)A_\nu \geq 24 - 4A_6 = 16, \tag{24}$$

which requires at least six regions of more than six sides, and hence by (13) at least 18 pentagons. This would make 26 regions. If the map contained less than two hexagons, the same equation would show that there were more than 27 regions in the map, by a similar argument. Also our map can not have less than five regions of more than six sides, since if the map contained four heptagons, (23) would give:

$$3A_6 > 16, \tag{25}$$

and the six hexagons required by (25) together with the 16 pentagons required by (13) would make 26 regions. Furthermore each heptagon less than four will increase the right member of (25) by eight, and hence require at least two additional hexagons in place of the heptagon and pentagon removed. Replacing any of the heptagons by octagons or regions of more than eight sides will strengthen our inequalities, as well as necessitating more pentagons to satisfy (13).

But the map of 17 pentagons, three hexagons, and five heptagons is not irreducible, since it does not satisfy (21). Consequently every irreducible map must contain more than 25 regions and this gives the theorem:

Every map containing 25 *or fewer regions can be colored in four colors.*

6. The question naturally arises whether 25 is the greatest number for which we can prove such a theorem as the above on the basis of the reductions already described. While an exact answer to this question is lacking, it is evident that the smallest number of regions in a map not containing any of these known reducible configurations is not considerably above 25, as we can construct a map with a small number of regions not containing any of them. Thus in Fig. 7 we exhibit a map of 42 regions which satisfies all the properties of irreducible maps given by previous writers as well as those derived in this paper. The map may be formed by constructing a hexagon on each of the 30 edges of a regular dodecahedron, in such a way

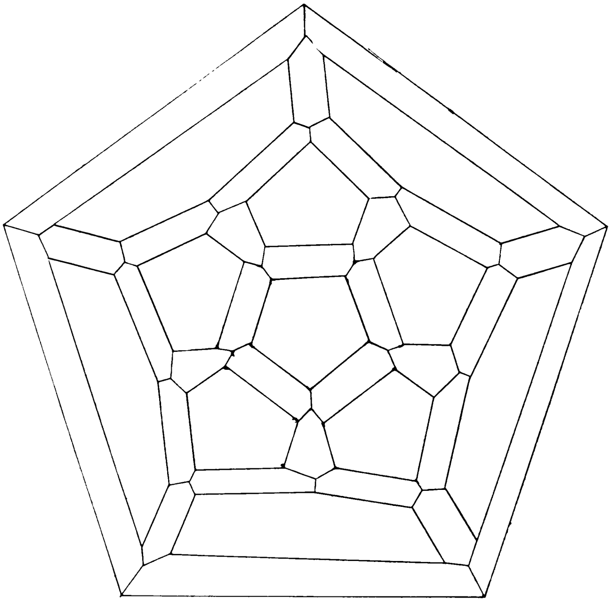

FIG. 7.

as to leave twelve pentagonal faces,* and may be colored by first coloring the pentagons as they would be colored for the dodecahedron.

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* This process was suggested to the writer in another connection by Prof. J. W. Alexander.